Radiation reaction reexamined: bound momentum and Schott term

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We review and compare two different approaches to radiation reaction in classical electrodynamics of point charges: a local calculation of the self-force using the charge equation of motion and a global calculation consisting in integration of the electromagnetic energy-momentum flux through a hypersurface encircling the world-line. Both approaches are complementary and, being combined together, give rise to an identity relating the locally and globally computed forces. From this identity it follows that the Schott terms in the Abraham force should arise from the bound field momentum and can not be introduced by hand as an additional term in the mechanical momentum of an accelerated charge. This is in perfect agreement with the results of Dirac and Teitelboim, but disagrees with the recent calculation of the bound momentum in the retarded coordinates. We perform an independent calculation of the bound electromagnetic momentum and verify explicitly that the Schott term is the derivative of the finite part of the bound momentum indeed. The failure to obtain the same result using the method of retarded coordinates tentatively lies in an inappropriate choice of the integration surface. We also discuss the definition of the delta-function on the semi-axis involved in the local calculation of the radiation reaction force and demonstrate inconsistency of one recent proposal.

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I. INTRODUCTION AND OVERVIEW

Studies of the radiation reaction in classical electrodynamics initiated by Lorentz and Abraham as far as in the 19-th century, remained an area of active research during the whole 20-th century. Although with the development of quantum electrodynamics this problem became somewhat academic, it still attracts attention in connection with new applications and new ideas in fundamental theory. The current understanding of the radiation reaction has emerged in the classical works of Dirac [1], Ivanenko and Sokolov [2], Rohrlich [3], Teitelboim [4] and some others. One of the crucial points of this theory is the nature of the non-dissipative term in the reaction force known as the Schott term [5]. According to Dirac [1] and Teitelboim [4], this term is a finite part of the derivative of the momentum of the electromagnetic field which is bound to the charge. Meanwhile, as far as we are aware, in the existing literature there are only two explicit calculations of the bound momentum via integration of the corresponding flux: that by Dirac and a more recent calculation by Poisson [6], who used the method of retarded coordinates of Newman and Unti [7]. The results disagree, namely, according to [6], the bound momentum produces only an infinite term which has to be absorbed by the mass renormalization. The purpose of this paper is to clarify the nature of the above discrepancy and to provide an independent calculation of the bound momentum.

In the famous paper published in 1938, Dirac [1] gave a detailed and consistent derivation of the equation of motion of a point radiating charge in four-dimensional flat space-time. In his original formulation, the decomposition of the retarded field into the

sum of the radiation field (the half-difference of the retarded and advanced fields) and the self-action field (the half-sum of the same quantities) was suggested. When such a decomposition is used to describe the action of the proper field on the charge itself, the first part provides the finite radiation-reaction force (the Abraham vector [8])

$$f_{\rm rad}^\mu = \frac{2}{3}e^2(a^2 v^\mu + \dot{a}^\mu),\qquad(1)$$

where v^μ is the four-velocity and $a^\mu = \dot{v}^\mu$ is the four-acceleration. Here the dot denotes the derivative with respect to the proper time, $a^2 = a_\mu a^\mu$, and the signature $+,-,-,-$ is understood. The second part contributes only to an infinite renormalization of mass and does not affect the motion of the charge in the flat space-time.

The reaction force contains the radiation recoil term, the first term in (1),

$$f_{\rm emit}^\mu = \frac{2}{3}e^2 a^2 v^\mu,\qquad(2)$$

and the Schott term

$$f_{\rm Schott}^\mu = \frac{2}{3}e^2 \dot{a}^\mu.\qquad(3)$$

This second term is a total derivative, so it does not correspond to an irreversible loss of momentum by the particle, but plays an important role in the momentum balance between the radiation and particle momentum loss. If $f_{\rm rad}^\mu = 0$, this does not necessarily mean that there is no radiation (recall the well-known case of radiation from a uniformly accelerated charge [2, 9]), but if there is no radiation, the Abraham force (1) is zero. Indeed, if one has $a_\mu a^\mu = 0$ at any time,

then it is easy to show that the three-acceleration is zero, $\mathbf{a} = 0$, and therefore $\dot{a}^\mu = 0$. Thus, no radiation reaction force is possible in the absence of radiation.

The correct interpretation and a constructive derivation of the Schott term was given by Dirac himself [1] via integration of the Maxwell stress-tensor of the charge retarded field over the space-like hypersurface orthogonal to the world-line. The subsequent discussion was somehow obscured by the use in this paper of the advanced potential. But Havas [10] noticed that actually only the physical retarded field was involved in the calculation of the bound electromagnetic momentum, and the whole expression for the Abraham vector can be obtained using only the retarded field [11]. This became especially transparent after a later investigation of the nature of the Schott term by Teitelboim [4] (see also the review [12]), where it was emphasized that this term originates from the bound electromagnetic momentum. However, having provided a very clear and comprehensive discussion, Teitelboim did not present an explicit calculation of the quantities involved, addressing the reader to the Dirac's paper for technical details. Meanwhile, the details of the integration carried out in [1] are fairly nontrivial, and that is why more recently an attempt was made in Ref. [6] to simplify the derivation using the retarded coordinates of Newman and Unti [7]. This simplified method seems to be getting popular, and it has been generalized to arbitrary dimensions [13]. However, this modified calculation fails to give the Schott term as a part of the bound electromagnetic momentum. This has led the author of [6] to revive the attempt to ascribe a *mechanical* origin to the Schott term, which interpretation can in fact be found in the earlier literature.

The crucial point in Dirac's calculation was the power series expansion of the retarded field in terms of the suitably defined small parameter related to the proper time difference between the moments of the emission and observation. This is necessary because the integrand of the associated integral expression contains retarded fields taken at different moments of the proper time. Although one might think that such expansions can be avoided by using the retarded coordinated, a more careful analysis shows that this is not so. We present here a straightforward calculation which is technically slightly different from that used by Dirac, but is similar conceptually. We have also generalized the calculation to arbitrary space-time dimensions (to be presented elsewhere), which sheds new light to the problem of the Schott term. Note that multidimensional generalization of the Maxwell theory was discussed by Ivanenko and Sokolov [14], soon after the work of Dirac, in connection with the Huygens principle, and recently this problem has attracted attention in view of general interest to space-time models with large extra dimensions [13, 15, 16, 17, 18]).

Some confusion about the Schott term is also related to the well-known 'phenomenological' derivation of the Lorentz-Dirac equation, as given in the book by Landau and Lifshitz [19]. In this heuristic derivation, the first term of the Abraham force is obtained by computing the rate of radiation, while the Schott term is added by hand from the requirement of orthogonality of the reaction force to the particle four-velocity. Formally, this procedure leads to the correct equation (though does not answer the question about the physical origin of the Schott term), so *per se* it does not contradict to the correct interpretation according to which the Schott term is viewed as a part of the bound momentum. But as we show here, if one does not relate the Schott term to the electromagnetic momentum, the energy-momentum balance equations become contradictory (Sect. II). The essential difference between the 'phenomenological' derivation of the Schott term via the orthogonalization procedure and its consistent treatment as the derivative of the finite part of the bound electromagnetic momentum becomes especially clear in higher dimensions. It turns out that generically the number of possible momentum 'counterterms' in higher dimensions is larger than the number of equations arising from the requirement of the orthogonality. As a result, the Schott term(s) can not be obtained within the orthogonalization procedure in even dimensions higher than six.

The redefinition of the mechanical momentum of a radiating charge would also be wrong conceptually, since it would imply that the Maxwell-Lorentz electrodynamics *be not* muliplicatively renormalizable: a new (finite) mechanical term not present in the initial lagrangian would be required. In fact, failure of the multiplicative renormalizability is what happens indeed in higher-dimensional electrodynamics [16, 17], but an essential property of the *four-dimensional* theory is the renormalizability in the sense that no additional counterterms (either infinite or finite) are required to make the theory consistent. This reflects the renormalizability of *quantum* electrodynamics in four dimensions.

Our other remark concerns the regularization of products of the delta-function and its derivatives with the Heaviside function which arise in the quasilocal treatment of the radiation reaction problem. Recently this problem was reconsidered in [17] in the context of the higher-dimensional generalization of the Maxwell electrodynamics, where a new regularization scheme was suggested and used to derive the Lorentz-Dirac equation in even dimensions higher than four. We will show below that the regularization proposed in [17] fails to reproduce the correct result already in the case of four dimensions, so the validity of the equations derived in [17, 18] (and of the regularization itself) is questionable. Meanwhile the consistent treatment of the delta-function on a half-line via the point-splitting was suggested long ago (see e.g. the books by Ivanenko and Sokolov [2], Rohrlich [3] and Barut [20]), and we have checked that it works per-

fectly well in any even space-time dimensions. The actual check of validity of any regularization of this kind, as was emphasized by Dirac, consists in an alternative calculation of the reaction force by integrating the rate of the variation of the bound electromagnetic momentum.

II. FIELD DECOMPOSITION VERSUS STRESS TENSOR DECOMPOSITION

Our definitions closely follow those by Rohrlich [3] and Teitelboim [4]. The retarded potential generated by a point charge moving along the world-line $x^\mu = z^\mu(s)$ depends on the kinematic variables taken at the (position dependent) retarded proper time $s_{\rm ret}(x)$ defined as the solution to the equation

$$R^\mu R_\mu = 0, \quad R^\mu = x^\mu - z^\mu(s_{\rm ret}), \qquad (4)$$

satisfying $x^0 > z^0$. The second solution to the same equation with $z^0 > x^0$ defines the advanced proper time $s_{\rm adv}(x)$. Introducing the invariant distance

$$\rho = v_\mu(s_{\rm ret})R^\mu, \quad v^\mu = \frac{dz^\mu}{ds}, \qquad (5)$$

which is equal to the spatial distance $|\mathbf{R}| = |\mathbf{x} - \mathbf{z}(s_{\rm ret})|$ between the points of emission and observation in the momentarily co-moving Lorenz frame at the time moment $x^0 = z^0(s_{\rm ret})$, one can present the retarded potential as

$$A_{\rm ret}^\mu(x) = \left.\frac{ev^\mu}{\rho}\right|_{s_{\rm ret}(x)}. \qquad (6)$$

It is convenient to introduce the null vector $c^\mu = R^\mu/\rho$, whose scalar product with v^μ is equal to unity, and also the unit space-like vector $u^\mu = c^\mu - v^\mu$. Thus we have

$$v^2 = 1, \quad c^2 = 0, \quad vc = 1, \quad u^2 = -1. \qquad (7)$$

(Here and below we omit, where unambiguous, brackets in the four-dimensional scalar products, e.g. $vc = v^\mu c_\mu$). Differentiations with respect to x^μ are performed using the relations

$$\begin{aligned}
\partial_\mu s_{\rm ret}(x) &= c_\mu, \\
\partial_\mu \rho &= v_\mu + \lambda c_\mu, \\
\partial_\mu c^\nu &= \frac{1}{\rho}\left(\delta_\mu^\nu - v_\mu c^\nu - c_\mu v^\nu - \lambda c_\mu c^\nu,\right)
\end{aligned} \qquad (8)$$

where

$$\lambda = \dot\rho = \rho(ac) - 1. \qquad (9)$$

Using these formulas we obtain the field tensor

$$F_{\mu\nu} = \frac{e\,(\rho(ac) - 1)}{\rho^2}v_{[\mu}c_{\nu]} - \frac{e}{\rho}a_{[\mu}c_{\nu]} \qquad (10)$$

where square brackets denote antisymmetrization without factor 1/2, e.g. $a_{[\mu}c_{\nu]} = a_\mu c_\nu - a_\nu c_\mu$, and all quantities have to be taken at the moment $s_{\rm ret}(x)$ of the proper time . Similarly, the advanced potential $A_{\rm adv}^\mu$ and the corresponding field can be written in terms of quantities depending on $s_{\rm adv}(x)$.

Separation of the radiation from the total field can be performed in two different ways. The first procedure is linear in the field and consists in the splitting the retarded potential $A_{\rm ret}^\mu$ into the radiative part

$$A_{\rm rad}^\mu = \frac{1}{2}\left(A_{\rm ret}^\mu - A_{\rm adv}^\mu\right) \qquad (11)$$

and the 'self' part

$$A_{\rm self}^\mu = \frac{1}{2}\left(A_{\rm ret}^\mu + A_{\rm adv}^\mu\right). \qquad (12)$$

The radiative potential satisfies the homogeneous D'Alembert equation and changes the sign under reflection of time, as expected for the radiation irreversibly lost by an accelerated charge. This part of the retarded field tends to zero in the static limit. The self part is time-symmetric and remains finite in the static limit, where it coincides with the Coulomb potential.

The split of the second kind is quadratic in the field and uses the energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F_\lambda{}^\nu + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right). \qquad (13)$$

Considering the energy-momentum tensor we will always deal with the retarded solution of the D'Alembert equation. Substituting here the field tensor (10) we obtain the sum of two terms

$$T^{\mu\nu} = T_{\rm emit}^{\mu\nu} + T_{\rm bound}^{\mu\nu}, \qquad (14)$$

where the first term is proportional to ρ^{-2}:

$$\frac{4\pi}{e^2}T_{\rm emit}^{\mu\nu} = -\frac{(ac)^2 + a^2}{\rho^2}c^\mu c^\nu, \qquad (15)$$

while the second one contains higher powers of ρ^{-1}:

$$\begin{aligned}
\frac{4\pi}{e^2}T_{\rm bound}^{\mu\nu} &= \frac{a^{(\mu}c^{\nu)} + 2(ac)c^\mu c^\nu - (ac)v^{(\mu}c^{\nu)}}{\rho^3} \\
&+ \frac{v^{(\mu}c^{\nu)} - c^\mu c^\nu - \eta^{\mu\nu}/2}{\rho^4}.
\end{aligned} \qquad (16)$$

Here the symbol $(\mu\nu)$ denotes a symmetrization without the factor 1/2.

The first expression (15) is distinguished by the following properties:

- its geometric structure is the tensor product of two null vectors c^μ,

- it is traceless,

- it falls down as $|\mathbf{x}|^{-2}$ when $|\mathbf{x}| \to \infty$,

- it is independently conserved

$$\partial_\nu T^{\mu\nu}_{\text{emit}} = 0. \tag{17}$$

This latter property follows from the differentiation rules (8). All these features indicate that $T^{\mu\nu}_{\text{emit}}$ describes an outgoing radiation. An independent conservation of this quantity means that the bound part is also independently conserved

$$\partial_\nu T^{\mu\nu}_{\text{bound}} = 0. \tag{18}$$

Conservation of the total four-momentum implies that the sum of the mechanical momentum and the momentum of the electromagnetic field is constant (for simplicity we do not include an external field):

$$\frac{dp^\mu_{\text{mech}}}{ds} + \frac{dp^\mu_{\text{em}}}{ds} = 0. \tag{19}$$

Here the mechanical part is proportional to the bare mass of the charge

$$p^\mu_{\text{mech}} = m_0 v^\mu, \tag{20}$$

while the field part is given by

$$p^\mu_{\text{em}} = \int T^{\mu\nu} d\Sigma_\nu, \tag{21}$$

where integration of the electromagnetic stress tensor is performed over a space-like hypersurface whose choice will be discussed in detail later on. It has to be emphasized again that in the expression (13) for the stress tensor of the electromagnetic field one has to use the physical retarded field. According to the second splitting, one can write

$$\frac{dp^\mu_{\text{mech}}}{ds} = -\frac{dp^\mu_{\text{em}}}{ds} = f^\mu_{\text{emit}} + f^\mu_{\text{bound}}, \tag{22}$$

$$f^\mu_{\text{emit}} = -\frac{d}{ds}\int T^{\mu\nu}_{\text{emit}} d\Sigma_\nu, \tag{23}$$

$$f^\mu_{\text{bound}} = -\frac{d}{ds}\int T^{\mu\nu}_{\text{bound}} d\Sigma_\nu. \tag{24}$$

On the other hand, the derivative of the bare mechanical momentum can be expressed using the equation of motion of the charge in which the electromagnetic field is decomposed into the self part (12) and the radiation part (11)

$$\begin{aligned} \frac{dp^\mu_{\text{mech}}}{ds} &= e F^{\mu\nu}_{\text{ret}} v_\nu \\ &= e\left(F^{\mu\nu}_{\text{self}} + F^{\mu\nu}_{\text{rad}}\right) v_\nu \\ &= f^\mu_{\text{self}} + f^\mu_{\text{rad}}. \end{aligned} \tag{25}$$

Clearly, the following energy-momentum conservation identity should hold in view of (22) and (25):

$$f^\mu_{\text{self}} + f^\mu_{\text{rad}} = f^\mu_{\text{bound}} + f^\mu_{\text{emit}}. \tag{26}$$

Now, somewhat unexpectedly, $f^\mu_{\text{rad}} \neq f^\mu_{\text{emit}}$ and $f^\mu_{\text{self}} \neq f^\mu_{\text{bound}}$, differing by the Schott term:

$$f^\mu_{\text{rad}} = f^\mu_{\text{emit}} + f^\mu_{\text{Schott}}, \tag{27}$$

$$f^\mu_{\text{self}} = f^\mu_{\text{bound}} - f^\mu_{\text{Schott}}. \tag{28}$$

The identity (26) is satisfied as expected. Explicit calculations verifying these results will be presented in what follows. They are fully consistent with the results of Dirac [1], Rohrlich [3] and Teitelboim [4]. On the contrary, in [6] it was found that $f^\mu_{\text{bound}} = f^\mu_{\text{self}}$, while the Eq. (27) still holds. This is obviously inconsistent with the energy-momentum conservation identity (26).

To avoid any confusion, we note that though both sides of the energy-momentum conservation identity (26) contain divergent terms, the extraction of finite terms is fully unambiguous because of their different dependence on the kinematical variables. Moreover, the parameterization of the divergent terms can be made similar in both calculations, so these terms mutually cancel in (26) before the regularization is removed.

III. WORLD-LINE CALCULATION: POINT SPLITTING

The retarded and advanced potential taken on the world-line $x^\mu = z^\mu(s)$ of a charge can conveniently be written in terms of Green's functions [2]

$$G_{\text{self}}(Z) = \delta(Z^2), \tag{29}$$

$$G_{\text{rad}}(Z) = \frac{Z^0}{|Z^0|}\delta(Z^2), \tag{30}$$

where $Z^\mu = Z^\mu(s, s') = z^\mu(s) - z^\mu(s')$. Substitution of the electromagnetic field of the charge on its world-line leads to the following integrals

$$f^\mu(s) = 2e^2 \int Z^{[\mu}(s, s')v^{\nu]}(s')v_\nu(s)\frac{d}{dZ^2}G(Z)ds', \tag{31}$$

for both f^μ_{self} and f^μ_{rad}. Due to the presence of delta-functions in G_{self} and G_{rad}, one is tempted to expand the integrands in $\sigma = s - s'$. Taking into account that $Z^2 = \sigma^2 + O(\sigma^4)$, one can write

$$G_{\text{self}}(Z) = \delta(\sigma^2) + O(\sigma^4), \tag{32}$$

$$G_{\text{rad}}(Z) = \frac{\sigma}{|\sigma|}\left(\delta(\sigma^2) + O(\sigma^4)\right). \tag{33}$$

Expanding the rest of the integrands in σ, one encounters the following integrals:

$$A_l = \int_{-\infty}^{\infty} \sigma^l \frac{d}{d\sigma^2}\delta(\sigma^2)\, d\sigma, \tag{34}$$

for the self-force, and

$$B_l = \int_{-\infty}^{\infty} \sigma^l \frac{d}{d\sigma^2}\left(\frac{\sigma}{|\sigma|}\delta(\sigma^2)\right) d\sigma \tag{35}$$

for the radiation reaction force, with $l \geq 2$.

Both these integrals are ill-defined. Passing to the variable $x = \sigma^2$ they can be transformed to integrals of the type

$$\int_0^\infty \phi(x)\delta^{(k)}(x)dx, \qquad (36)$$

where $\delta^{(k)}(x)$ denotes the k-s derivative of the delta-function, which function has the support at the boundary point of the integration domain. Computing such an integral is equivalent to taking the product of the delta-function and the Heaviside function, or, equivalently, defining $\delta(x)$ on the semi-axis. For this a suitable regularization is needed. Before discussing this point, we note that, with any regularization, the integrals (34), (35) should vanish for $l > 3$ by power counting. Moreover, all A_l vanish for odd l, and all B_l vanish for even l from parity considerations. By power counting one can also show that terms in (32), (33) proportional to σ^4 actually give no contribution, while the relevant terms that must be retained in the expansion of the integrand are given by

$$2X^{[\mu}(s,s')v^{\nu]}(s')v_\nu(s) = a^\mu\sigma^2 - \frac{2}{3}(\dot{a}^\mu + v^\mu a^2)\sigma^3. \qquad (37)$$

Now we discuss the meaning of the delta-function on the semi-axis. Recently an attempt was made [17] to develop a general theory for such objects using primary definitions of the theory of distributions. The proposed regularization for the first derivative of the delta-function defined on the semi-axis (see Eq. (23) in [17]) reads:

$$\delta'(x) = \lim_{\alpha \to +0} \frac{\partial}{\partial \alpha} \frac{e^{-x/\alpha}}{\alpha}, \quad x \geq 0. \qquad (38)$$

Using this regularization one finds:

$$A_2 = \lim_{\alpha \to 0} \frac{\sqrt{\pi}}{4\sqrt{\alpha}}, \qquad (39)$$

$$B_3 = 1. \qquad (40)$$

Substituting this into the above formulas we obtain

$$f_{\text{self}}^\mu = \lim_{\alpha \to 0} \frac{e^2\sqrt{\pi}}{4\sqrt{\alpha}}a^\mu, \qquad (41)$$

$$f_{\text{rad}}^\mu = -\frac{2e^2}{3}(v^\mu a^2 + \dot{a}^\mu). \qquad (42)$$

The first quantity can be absorbed by the renormalization of mass, while the second gives the correct structure for the Abraham force but with the *wrong sign.* Thus the proposal [17] for the delta-function and its derivatives on the semi-axis fails to give the correct result.

Meanwhile, a satisfactory method to deal with such integrals was suggested long ago (see [2, 3, 20]), it consists in the 'point-splitting'

$$\delta(\sigma^2) = \lim_{\varepsilon \to +0} \delta(\sigma^2 - \varepsilon^2) = \lim_{\varepsilon \to +0} \frac{\delta(\sigma - \varepsilon) + \delta(\sigma + \varepsilon)}{2\varepsilon} \qquad (43)$$

with a prescription that the limit should be taken only after evaluating all the integrals. With this, omitting the symbol of the limit, we obtain

$$A_2 = -\frac{1}{2\varepsilon}, \qquad (44)$$

$$B_3 = -1, \qquad (45)$$

so one finds

$$f_{\text{self}}^\mu = -\frac{e^2}{2\varepsilon}a^\mu, \qquad (46)$$

$$f_{\text{rad}}^\mu = \frac{2e^2}{3}(v^\mu a^2 + \dot{a}^\mu). \qquad (47)$$

After the mass renormalization,

$$m_0 - A_2 = m, \qquad (48)$$

we get the Lorentz-Dirac equation

$$ma^\mu = \frac{2e^2}{3}(v^\mu a^2 + \dot{a}^\mu), \qquad (49)$$

One has to realize, however, that there is no purely mathematical proof of the correctness of the point splitting precedure. So actually to check the validity of this or any other regularization involved in the local calculation of the radiation reaction force one has to perform an alternative calculation via the integration of the electromagnetic stress-tensor.

IV. INTEGRATION OF ELECTROMAGNETIC MOMENTUM

A. General setting

An alternative derivation of the radiation reaction force consists in the integration of the energy-momentum tensor of the electromagnetic field of a charge over an appropriate hypersurface in space-time. The correct choice of this hypersurface is essential for the calculation, so we will discuss it in detail. Assuming the split (14) of the stress-tensor into the emitted and bound parts, one can consider the emitted and bound momenta separately. Teitelboim [4] defined the corresponding integration surfaces differently in both cases taking into account the specific properties of these quantities. Here we will show that both integrals can be transformed to those over an infinitely thin world tube around the particle world-line.

We would like to calculate the four-momentum carried by the electromagnetic field of the charge for a given moment of the proper time s on the particle world-line $z^\mu(s)$. To do this one has to choose a space-like hypersurface $\Sigma(s)$ intersecting the world-line at
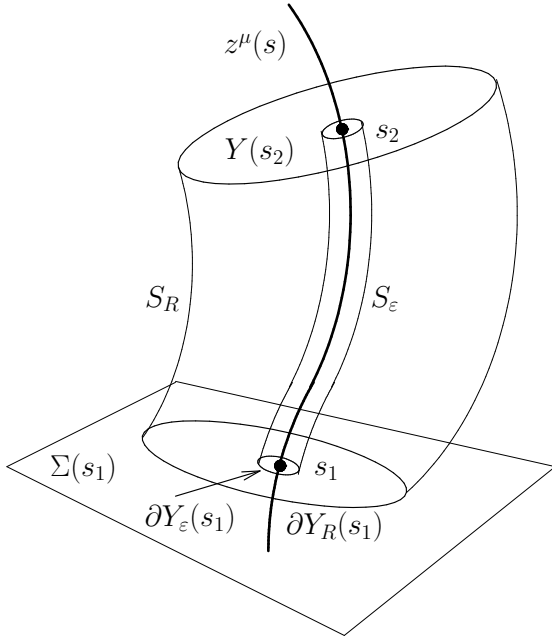
FIG. 1: Integration of the bound electromagnetic momentum. Here $\Sigma(s_1)$ is the space-like hyperplane transverse to the world-line $z^\mu(s)$ intersecting it at the proper time s_1 (similarly $\Sigma(s_2)$). The hypersurfaces S_ε and S_R are small and large tubes around the world-line formed by sequences of the 2-spheres $\partial Y_\varepsilon(s)$ and $\partial Y_R(s)$ for $s \in [s_1, s_2]$. The domain $Y(s_2) \subset \Sigma(s_2)$ (similarly $Y(s_1)$) is the 3-annulus between $\partial Y_R(s_2)$ and $\partial Y_\varepsilon(s_2)$.

the point $z^\mu(s)$ and to integrate the electromagnetic energy-momentum flux as follows

$$p_{\text{em}}^\mu(s) = \int_{\Sigma(s)} T^{\mu\nu} d\Sigma_\nu. \qquad (50)$$

The simplest and the most practical choice for $\Sigma(s)$ is that of the hypersurface orthogonal to the world-line

$$v_\mu(s) \left(x^\mu - z^\mu(s) \right) = 0. \qquad (51)$$

The integral (50) is divergent near the world line. The rate of the divergence can be controlled introducing the small length parameter ε, the radius of the 2-sphere $\partial Y_\varepsilon(s)$ (Fig. 1), defined by the intersection of the hyperplane (51) with the hyperboloid

$$(x - z(s))^2 = -\varepsilon^2. \qquad (52)$$

We also introduce the sphere $\partial Y_R(s)$ of a large radius R defined by the intersection of $\Sigma(s)$ with the hyperboloid

$$(x - z(s))^2 = -R^2. \qquad (53)$$

The electromagnetic momentum can then be obtained by taking the limit $\varepsilon \to 0$, $R \to \infty$ of the integral over the domain $Y(s) \subset \Sigma(s)$ between the boundaries $\partial Y_\varepsilon(s)$ and $\partial Y_R(s)$.

Let us evaluate the variation of this quantity between the moments s_1 and s_2 of the proper time on the world-line of the charge

$$\Delta p_{\text{em}}^\mu = \int_{Y(s_2)} T^{\mu\nu} d\Sigma_\nu - \int_{Y(s_1)} T^{\mu\nu} d\Sigma_\nu. \qquad (54)$$

For the bound momentum it is convenient to consider the tubes S_ε and S_R formed as sequences of the spheres $\partial Y_\varepsilon(s)$ and $\partial Y_R(s)$ on the interval $s \in [s_1, s_2]$ and to transform this quantity to

$$\Delta p_{\text{bound}}^\mu = \int_{S_R} T_{\text{bound}}^{\mu\nu} dS_\nu - \int_{S_\varepsilon} T_{\text{bound}}^{\mu\nu} dS_\nu \qquad (55)$$

in view of the conservation equation for $T_{\text{bound}}^{\mu\nu}$ (18). Here normal vectors in dS_ν are directed outwards with respect to the world-line. The contribution from the infinitely distant surface S_R vanishes if one assumes that the charge acceleration is zero in the limit $s \to -\infty$ [4]. This assertion is somewhat non-trivial, since, in spite of the fact that the stress tensor (16) decays as R^{-3}, the corresponding flux does not vanish a priori, because the surface element contains a term (proportional to the acceleration) which asymptotically grows as R^3 (see the Eq. 69 below). As a consequence, the surviving term will be proportional to the acceleration taken at the moment s_{ret} of the proper time, where $s_{\text{ret}} \to -\infty$ in the limit $R \to \infty$. Finally we are left with the integral over the inner boundary only

$$\Delta p_{\text{bound}}^\mu = -\int_{S_\varepsilon} T_{\text{bound}}^{\mu\nu} dS_\nu. \qquad (56)$$

For integration of the emitted momentum it is convenient to take the light cone boundary $C(s')$ instead of S_R as shown in Fig. 2. The actual change of the emitted momentum in the whole three-space corresponds to the limits $s' \to -\infty, \varepsilon \to 0$. Since the normal to the light cone lies on it, the flux of the energy-momentum tensor through the null boundary $C(s', s_1, s_2)$ is zero for any s', therefore

$$\Delta p_{\text{emit}}^\mu =$$

$$= \int_{Z(s',s_2)} T_{\text{emit}}^{\mu\nu} v_\nu(s_2) dS - \int_{Z(s',s_1)} T_{\text{emit}}^{\mu\nu} v_\nu(s_1) dS =$$

$$= -\int_{S_\varepsilon} T_{\text{emit}}^{\mu\nu} dS_\nu + \int_{C(s',s_2,s_1)} T_{\text{emit}}^{\mu\nu} c_\nu dS =$$

$$= -\int_{S_\varepsilon} T_{\text{emit}}^{\mu\nu} dS_\nu \qquad (57)$$
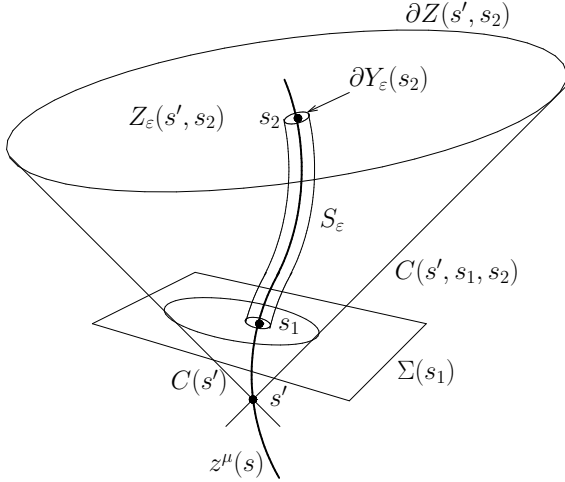
FIG. 2: Integration of the emitted momentum. Here $C(s')$ is the future light cone of some point s' on the world-line, and $C(s', s_1, s_2)$ is its part between the hypersurfaces $\Sigma(s_1)$ and $\Sigma(s_2)$. The domain $Z(s', s_2)$ (similarly $Z(s', s_1)$) is the annulus between the intersection of the light cone with $\Sigma(s_2)$ (the outer boundary) and the small sphere $\partial Y_\varepsilon(s_2)$ (the inner boundary).

where the limits $\varepsilon \to 0$, $s' \to -\infty$ have to be taken. So both the emitted and bound momenta can be reduced to integrals over the small tube around the world-line.

Now we have to find an integration measure on the small tube.

B. Induced metric on S_ε

Consider the foliation of the space-time region shown in Fig. 1 by the hypersurfaces $\Sigma(s)$ parameterized by the spherical coordinates $r, \theta_1 = \theta, \theta_2 = \varphi$. Introducing the unit space-like vector $n^\mu(s, \theta_i)$, $n_\mu n^\mu = -1$, transverse to v^μ, we can write the following coordinate transformation from x^μ to the set s, r, θ_i:

$$x^\mu = z^\mu(s) + rn^\mu(s, \theta_i), \qquad (58)$$
$$v_\mu(s)n^\mu(s, \theta_i) = 0, \qquad (59)$$
$$n_\mu(s, \theta_i)n^\mu(s, \theta_i) = -1. \qquad (60)$$

Since the 4-acceleration vector a^μ is orthogonal to the 4-velocity, it lies in the hyperplane $\Sigma(s)$ and we can further specify the angular coordinates choosing the polar axis along the three-acceleration $\mathbf{a}$. Our convention about the angle variables is such that the four-vector n^μ in the Lorentz frame momentarily comoving with the charge at the proper time moment s, CF(s), is given by

$$n_c^\mu = (0, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \qquad (61)$$

Then $a_\mu n^\mu = -a\cos\theta$, where $a = |\mathbf{a}|$. Differentiating (58) with respect to the new coordinates we obtain:

$$\frac{\partial x^\mu}{\partial s} = v^\mu(s) + r\frac{\partial n^\mu(s, \theta_i)}{\partial s}, \qquad (62)$$
$$\frac{\partial x^\mu}{\partial r} = n^\mu(s, \theta_i), \qquad (63)$$
$$\frac{\partial x^\mu}{\partial \theta_i} = r\frac{\partial n^\mu(s, \theta_i)}{\partial \theta_i}. \qquad (64)$$

Let us calculate the derivative $\partial n^\mu(s, \theta_i)/\partial s$ in CF(s). To find the variation $n^\mu(s + ds) - n_c^\mu$, we consider another Lorentz frame, CF($s + ds$), which is comoving with the charge at $s + ds$. These frames are related by the Lorentz boost with the velocity $|\mathbf{v}(s + ds) - \mathbf{v}(s)| = |\mathbf{a}(s)ds| = ads$. Performing the Lorentz transformations and taking into account that $\cos\theta$ is the same in both frames one finds $n^\mu(s + ds)$ in CF(s):

$$n^\mu(s + ds) = (a\cos\theta ds, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Hence in this frame

$$\frac{n^\mu(s + ds) - n^\mu(s)}{ds} = (a\cos\theta, 0, 0, 0),$$

which is proportional to the four-velocity in the same frame $v^\mu = (1,0,0,0)$. Thus in an arbitrary frame one has

$$\frac{\partial n^\mu}{\partial s} = a\cos\theta v^\mu = -(an)v^\mu. \qquad (65)$$

Denoting the angle derivatives as

$$e_i^\mu = r\partial n^\mu(s, \theta_i)/\partial \theta_i,$$

we find the relations

$$n_\mu e_j^\mu = r\frac{\partial n^2}{2\partial \theta_j} = 0,$$
$$v_\mu e_j^\mu = r\frac{\partial(vn)}{\partial \theta_j} - rn_\mu\frac{\partial v^\mu}{\partial \theta_j} = 0, \qquad (66)$$
$$n_\mu v^\mu = 0,$$

showing that the four vectors $\{n^\mu, v^\mu, e_j^\mu\}$ represent the space-time vierbein, with e_j^μ being the tangent vectors to the 2-sphere. The induced metric on the sphere is $g_{ij} = \eta_{\mu\nu}e_i^\mu e_j^\nu$, and for the four-dimensional induced metric we obtain

$$g_{\mu\nu} = \begin{pmatrix} [1 - r(an)]^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -g_{ij} \end{pmatrix}, \qquad (67)$$

and therefore

$$\det g = [1 - r(an)]^2 \det g_{ij}. \qquad (68)$$

Hence the area element on the hypersurface $r =$const S_ε will be given by

$$d^3\sigma = r^2[1 - r(an)]dsd\Omega, \qquad (69)$$

where $d\Omega = \sin\theta d\theta d\varphi$. Setting $r = \varepsilon$ one finds for the induced metric on S_ε

$$d^3\sigma = \varepsilon^2[1 - \varepsilon(an)]dsd\Omega, \qquad (70)$$

From (66) one can see that n^μ is a unit vector normal to the tube, so finally

$$dS_\mu = \varepsilon^2[1 - \varepsilon(an)]n_\mu dsd\Omega. \qquad (71)$$

Substituting this into (57,56) we find for the derivative of the electromagnetic momentum the following expression

$$\frac{dp^\mu_{\text{em}}}{ds} = -\int \varepsilon^2[1 - \varepsilon(an)]T^{\mu\nu}_{\text{em}}n_\nu d\Omega, \qquad (72)$$

valid for both the emitted and bound parts, where the limit $\varepsilon \to 0$ is understood. It has to be realized that the dependence of $T^{\mu\nu}_{\text{em}}$ on ε is somewhat non-trivial. In fact, the energy-momentum tensor depends on the space-time point x^μ through the quantity ρ, depending directly on x^μ, and also through the retarded proper time s_{ret}. At the same time, we need to express the resulting quantity as a function of the proper time s corresponding to the intersection of the world-line with the space-like hypersurface. Carefully keeping track of all this we expand the stress-tensor in terms of ε as follows

$$T_{\mu\nu}(\rho, s_{\text{ret}})\big|_{S_\varepsilon} = \sum_{k=-4}^{\infty} \varepsilon^k \Theta^k_{\mu\nu}(s, n^\mu, v^\mu, a^\mu, \dot{a}^\mu).$$

The leading divergent terms here are proportional to ε^{-2} for the emitted part (15,) and to ε^{-4} for the bound part (16). Substituting this expansion into (72) we have to perform integration over the angles and then to take the limit $\varepsilon \to 0$. The integration is easily done using the formula

$$\int n_\mu n_\nu d\Omega = \frac{4\pi}{3}\Delta_{\mu\nu}, \quad \Delta_{\mu\nu} = v_\mu v_\nu - g_{\mu\nu}, \quad (73)$$

while the integration of products of an odd number of n_μ gives zero.

C. Emitted momentum

Considering first the emitted momentum case, we see from Eq. (15) that the area factor ε^2 compensates the denominator $\rho^2(\varepsilon) \sim \varepsilon^2$ in the expression for $T^{\mu\nu}_{\text{emit}}$, so in the limit $\varepsilon \to 0$ it is sufficient to take only the leading terms in the numerator

$$a^2|_{s_{\text{ret}}} = a^2,$$
$$c^\mu|_{s_{\text{ret}}} = n^\mu + v^\mu, \qquad (74)$$
$$(ac)^2|_{s_{\text{ret}}} = (an)^2, \qquad (75)$$

where all quantities in the right hand sides are taken at the proper time moment s. Omitting also the ε-term in the integration measure in (57) and assuming $s_1 = -\infty, s_2 = s$ we obtain

$$p^\mu_{\text{emit}}(s) = -\frac{e^2}{4\pi}\int_{-\infty}^{s} ds' \left(a^2 + (an)^2\right)(v^\mu + n^\mu)d\Omega. \qquad (76)$$

After integration over the angles we arrive at

$$p^\mu_{\text{emit}}(s) = -\frac{2e^2}{3}\int_{-\infty}^{s} ds' a^2 v^\mu(s'), \qquad (77)$$

so the 'emitted' contribution to the radiative force is

$$f^\mu_{\text{emit}} = -\frac{dp^\mu_{\text{emit}}}{ds} = \frac{2e^2}{3}a^2 v^\mu. \qquad (78)$$

D. Bound momentum

In this case calculations are substantially more involved. All quantities in (16) depend on the retarded time, and to facilitate their expansion in ε-series it is useful to express $T^{\text{bound}}_{\mu\nu}$ through the null vector $R^\mu = c^\mu \rho$:

$$\frac{4\pi}{e^2}T^{\mu\nu}_{\text{bound}} = \frac{a^{(\mu}R^{\nu)}}{\rho^4} + \frac{(2(aR) - 1)R^\mu R^\nu}{\rho^6} + $$
$$+ \frac{(1 - (aR))v^{(\mu}R^{\nu)}}{\rho^5} - \frac{\eta^{\mu\nu}}{2\rho^4}. \qquad (79)$$

The expansion of R^μ reads

$$R^\mu = x^\mu - z^\mu(s_{\text{ret}}) = x^\mu - z^\mu(s) + z^\mu(s) - z^\mu(s_{\text{ret}}) = $$
$$= \varepsilon n^\mu + v^\mu \sigma - \frac{1}{2}a^\mu \sigma^2 + \frac{1}{6}\dot{a}^\mu \sigma^3 + \mathcal{O}(\sigma^4), \qquad (80)$$

where $\sigma = s - s_{\text{ret}} > 0$ and all the vectors in the last line are taken at s. This is an expansion in powers of σ, but we need an expansion in powers of ε. The relation between the two can be found from the condition $R^2 = 0$. Assuming an expansion of σ in terms of ε

$$\sigma = \sum_k b_k \varepsilon^k, \qquad (81)$$

and substituting it into the equation $R^2 = 0$, one finds order by order the coefficients: $b_0 = 0, b_1 = 1, b_2 = an/2$ and so on. Thus, up to the third order terms, which is sufficient for our purposes, we obtain:

$$\sigma = \varepsilon + \frac{an}{2}\varepsilon^2 + \left(9(an)^2 + a^2 - 4\dot{a}n\right)\frac{\varepsilon^3}{24}. \qquad (82)$$

Substituting this into Eq. (80) we find:

$$R^\mu = (n^\mu + v^\mu)\varepsilon + ((an)v^\mu - a^\mu)\frac{\varepsilon^2}{2} + \qquad (83)$$
$$+ \left[\left(9(an)^2 + a^2 - 4\dot{a}n\right)v^\mu - 12(an)a^\mu + 4\dot{a}^\mu\right]\frac{\varepsilon^3}{24}.$$

Similar expansions for the velocity and the acceleration taken at the moment $s_{\rm ret}$ are

$$v^\mu|_{s_{\rm ret}} = v^\mu - a^\mu \varepsilon + (\dot a^\mu - (an)a^\mu)\frac{\varepsilon^2}{2} - \tag{84}$$

$$- \left[\left(9(an)^2 + a^2 - 4\dot an\right)a^\mu - 12(an)\dot a^\mu + 4\ddot a^\mu\right]\frac{\varepsilon^3}{24},$$

$$a^\mu|_{s_{\rm ret}} = a^\mu - \dot a^\mu \varepsilon + (\ddot a^\mu - (an)\dot a^\mu)\frac{\varepsilon^2}{2}. \tag{85}$$

The invariant distance parameter $\rho = v_\mu(s_{\rm ret})R^\mu$ is given by the product of the two expansions:

$$\rho = \varepsilon - an\frac{\varepsilon^2}{2} + \left(8(\dot an) - 3a^2 - 3(an)^2\right)\frac{\varepsilon^3}{24}. \tag{86}$$

For convenience we give also the expansion of the most singular term in (79) up to the relevant order:

$$\frac{1}{\rho^6} = \frac{1}{\varepsilon^6}\left[1 + 3(an)\varepsilon + \right.$$
$$\left. + \left(6(an)^2 + 3a^2/4 - 2\dot an\right)\varepsilon^2\right], \tag{87}$$

and the expansion of the scalar product (aR):

$$aR = an + \left(a^2/2 - \dot an\right)\varepsilon + \left[4\dot aa/3 + a^2(an) + \right.$$
$$\left. +\ddot av + \ddot an - (an)(\dot an)\right]\varepsilon^2/2. \tag{88}$$

Substituting all these expansions into (56) we obtain

$$\Delta p^\mu_{\rm bound} = \frac{e^2}{4\pi}\int_{s_1}^{s_2} ds \left\{ \frac{-n^\mu}{2\varepsilon^2} + \frac{a^\mu}{2\varepsilon} + \left[\left((an)^2 + a^2/3\right)v^\mu + \right.\right.$$
$$\left.\left. + \left((an)^2 + a^2/2\right)n^\mu - 2\dot a^\mu/3 + 3(an)a^\mu/4\right]\right\}d\Omega. \tag{89}$$

Using the above rules of the integration over the angles (73), one can see that the leading divergent term proportional to $1/\varepsilon^2$ vanishes and the result reads

$$\Delta p^\mu_{\rm bound} = e^2 \int_{s_1}^{s_2} ds \left(\frac{1}{2\varepsilon}a^\mu - \frac{2}{3}\dot a^\mu\right). \tag{90}$$

Setting $s_1 = -\infty, s_2 = s$, we obtain

$$p^\mu_{\rm bound} = e^2 \int_{-\infty}^{s} ds' \left(\frac{1}{2\varepsilon}a^\mu - \frac{2}{3}\dot a^\mu\right). \tag{91}$$

Therefore the bound part of the self-force is

$$f^\mu_{\rm bound} = -\frac{dp^\mu_{\rm bound}}{ds} = -\frac{e^2 a^\mu}{2\varepsilon} + \frac{2e^2}{3}\dot a^\mu. \tag{92}$$

Here the first divergent term has to be absorbed by the renormalization of mass, while the second is the Schott term. Comparing this with (46) we confirm the relation (28) and the identity (26). The Schott term therefore is the finite part of the derivative of the bound electromagnetic momentum of a charge. Note that *a priori* the parameter of regularization ε (the

radius of the small tube) is not related to the splitting parameter of the delta-function in the local force calculation. But actually they give the same form for the divergent term, for which reason we denoted them similarly. With this convention, the divergent terms in the momentum conservation identity (26) mutually cancel.

V. CONCLUSION

The goal of this paper was to clarify the recent discrepancy in calculations of the bound momentum of the radiating charge and to present an independent explicit calculation revealing the nature of the Schott term in the Lorentz-Dirac equation. We would like to emphasize the role of the global momentum conservation in understanding the origin of the Schott term. From the momentum conservation one can derive an identity relating the contributions to the reaction force in two alternative calculations: a local computation on the world-line and the global integration of the Maxwell stress-tensor. This identity demands that the Schott term *should* arise from the bound field momentum, and this is confirmed by an explicit calculation. Although this problem was discussed by a number of authors in the past, only few of them presented the details of calculation of the bound momentum, most notably Dirac [1], with whom we perfectly agree. We disagree, however, with more recent results of Poisson [6], who used a different method, based on retarded coordinated, in order to avoid somewhat lengthy expansions used in the Dirac's calculation. According to [6], the bound momentum reduces entirely to the mass-renormalization term. As we have shown here, this result is incompatible with the momentum conservation identity and thus can not be correct. More detailed analysis shows that the integration surface in [6] does not correspond to the definition of the bound momentum associated with the given moment of the proper time of a charge (see the Appendix below). Therefore we confirm the result of Dirac and show that no alternative interpretation of the Schott term is needed (and not possible, moreover). Technically, our derivation is slightly different from that of Dirac in performing all the necessary expansions in a uniform way. Conceptually, we fully agree with Teitelboim [4], presenting in addition full details of the global derivation of the Schott term.

Notice that in the literature one often encounters a confusing notation related to two alternative ways to select radiation. Namely, one has to distinguish the 'radiative' field as the half-difference of the retarded and advanced fields and the 'radiative' component of the Maxwell stress-tensor, which is defined through the retarded field. Here we suggest a different wording for the radiative part of the stress tensor, which we call 'emitted'. This distinction is necessary indeed since the corresponding local forces differ by

the Schott term.

Physically, the Schott term describes the reversible
variation of the Coulomb field bound momentum dur-
ing an accelerated motion of the charge. Variation
of the *mechanical* momentum of the charge consists
of this reversible part (which may be both positive
or negative) and an irreversible loss due to radiation.
In other words, the momentum radiated away can be
borrowed both from the mechanical momentum and
from the bound Coulomb momentum, and this ex-
plains how in the threshold case of the uniform ac-
celeration the mechanical momentum of the radiating
charge may remain constant. One is not allowed to
redefine the mechanical momentum of the charge by
adding to it the Schott term without facing contradic-
tion with the global energy-momentum conservation
identity.

A. Appendix. Integration of flux in retarded coordinates

Here for comparison we reproduce the formulation
of the problem in the retarded coordinates [6] (re-
cently generalized to arbitrary dimensions in [13]).
The main difference with our procedure is the defi-
nition of the integration hypersurface for the Maxwell
stress-tensor in terms of the retarded coordinates due
to Newman and Unti [7] ($s = s_{\rm ret}, \rho, \theta_i$). These are in-
troduced as follows. Selecting an arbitrary point $z(s)$
on the world-line one constructs its future light cone
(Fig. 4)

$$C(s) = \{x|(x - z(s))^2 = 0, x^0 \geqslant z^0\}, \quad (93)$$

ascribing to it the unique value of the coordinate s,
and defines on it the coordinate ρ as an affine length
parameter on a null geodesic specified by the angle
coordinates θ_i. The orbit of the constant $\rho(s, x)$ on the
cone (93) forms a two-dimensional manifold $\Sigma(s, \rho)$
which is an intersection of the hyperplane

$$H(s, \rho) = \{x|(x - z(s))^\mu v_\mu(s) = \rho = {\rm const}\} \quad (94)$$

and the cone (93). The open tube surrounding the
world-line is then defined as the sequence $\Sigma(u, \rho)$ of
hypersurfaces $\rho = {\rm const}$ for $u \in (-\infty, s]$. The area
element on this tube is equal to

$$dS_\mu = \rho^2(v_\mu(u) + \lambda(u, x)c_\mu(u, x))dud\Omega, \quad (95)$$
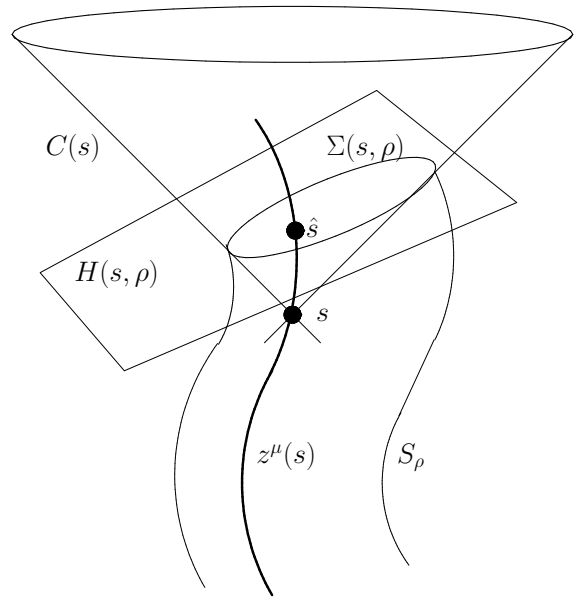

FIG. 3: Integration tube in the retarded coordinates. Here
$C(s)$ is a future light cone with an apex at $z(s)$, $H(s, \rho)$ is
a hyperplane transverse to $v(s)$ which intersects the world-
line at a point $\hat{s}$. The two-dimensional surface $\Sigma(s, \rho)$ is
an intersection $\Sigma(s, \rho) = C(s) \bigcap H(s, \rho)$, S_ρ is a lateral
hypersurface of constant ρ, formed by all $\Sigma(s', \rho)$, $s' = -\infty...s$.

where $c_\mu = (x - z(u))_\mu/\rho$. The factor ρ here is the
same as in the expressions (15), (16), so it partially
compensates the denominator of $T^{\mu\nu}_{\rm bound}$ and fully com-
pensates that of $T^{\mu\nu}_{\rm emit}$, which looks as simplification.
The procedure proposed in [6] consists in the inte-
gration of the flux over the tube for fixed ρ with the
subsequent limit $\rho \to 0$.

However, this procedure has a serious drawback
when applied to the bound part of the electromag-
netic momentum because of the singular nature of the
integrand at $\rho = 0$. In fact, this means that we have
to consider the sequence of tubes of variable radius ρ.
But with variable ρ the integral

$$\int_{-\infty}^{s} du \int T^{\mu\nu}\rho^2(v_\mu + (\rho(ac) - 1)c_\mu)d\Omega \quad (96)$$

does not give the field momentum associated with any
given moment of the proper time. Indeed, when ρ is
changing, the spheres $\Sigma(s, \rho)$ move across the light
cone and do not lie on a definite space-like hypersur-
face. For any fixed finite ρ the integral (96) is per-
formed over the hyperplane intersecting the world-line
at the point of proper time

$$\hat{s} = s + \tau(s, \rho) > s, \quad (97)$$

as shown on Fig. 3. Only in the limit $\hat{s} \to s$ one actually integrates over the hyperplane intersecting the world-line at s, but performing the limit as $\rho \to 0$ one passes through the sequence of *different* space-like hyperplanes. Contrary to this, our procedure consists in fixing the unique space-like hyperplane and carefully expanding the integrand in powers of the radius of the tube. This allows to pass consistently to the limit of the vanishing tube radius.

Therefore it is not surprising that a calculation along the lines of [6] gives for the bound momentum

only the leading divergent term (see the Eq. (8.3) of [6])

$$\frac{dp^\mu_{\text{bound}}}{ds} = \frac{e^2 a^\mu}{2\varepsilon}, \tag{98}$$

but fails to produce the second finite term in (92). For the emitted momentum the procedure of [6] works, since the answer in this case is given entirely by the leading term.

[1] P. Dirac, Proc. Roy. Soc. (London) **A167** (1938) 148.
[2] D. Ivanenko and A. Sokolov, *Klassicheskaya teorya polya*, Moscow 1948; *Klassische Feldtheorie*, Berlin 1953.
[3] F. Rohrlich, *Classical charged particles*, Addison-Wesley, Reading, Mass. 1965; 2-nd edition: Redwood City, CA 1990).
[4] C. Teitelboim, Phys. Rev. **D1** (1970) 1572.
[5] G. A. Schott, Phil. Mag. **29** (1915) 49.
[6] E. Poisson, *An introduction to the Lorentz-Dirac equation*, gr-qc/9912045.
[7] E. Newman and T. Unti, Journ. Math. Phys. **4** (1963) 1467.
[8] M. Abraham, *Theorie der Elektrizität*, vol. II, Springer, Leipzig, 1905.
[9] T. Fulton and F. Rohrlich, Ann. Phys. **9** (1960), 499.
[10] P. Havas, Phys. Rev. **74** (1948) 456.
[11] F. Rohrlich, Nuovo Cimento **21** (1961) 811.
[12] C. Teitelboim, D. Villarroel and Ch. G. Van Weert, Riv. Nuovo Cim. **3** (1980) 1-64.
[13] Yu. Yaremko, J. Phys. A: Math. Gen. **37** (2004) 1079.
[14] D. D. Ivanenko and A. A. Sokolov, Sov. Phys. Doklady, **36** (1940) 37.
[15] B. P. Kosyakov, Theor. Math. Phys. **199** (1999) 493.
[16] D. V. Gal'tsov, Phys. Rev. **D66** (2002) 025016; hep-th/0112110.
[17] P. O. Kazinski, S. L. Lyakhovich and A. A. Sharapov, Phys. Rev. **D66** (2002) 025017; hep-th/0201046.
[18] P. O. Kazinski and A. A. Sharapov, hep-th/0212286.
[19] L. D. Landau and E. M. Lifshitz, *The classical theory of fields*, Addison, Reading, Mass., 1962.
[20] A. O. Barut, *Electrodynamics and classical theory of fields and particles*, New York, 1964.